Compactness and Measures of Noncompactness in Metric Trees

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ABSTRACT.

A metric tree (M, d), also known as $\mathbb{R}$-trees or T-theory, is a metric space such that between any two points there is an unique arc and that arc is isometric to an interval in $\mathbb{R}$.

In this paper after presenting some fundamental properties of metric trees and metric segments, we will give a characterization of compact metric trees in terms of metric segments. Two common measures of noncompactness are the ball and set measures, respectively defined as

$$\beta(S) = \inf\{\epsilon \mid S \text{ has a finite } \epsilon\text{-net in } M\} \text{ and}$$

$$\alpha(S) = \inf\{\epsilon \mid S \text{ has a finite cover of sets of diameter } \leq \epsilon\}.$$

We will prove that $\alpha = 2\beta$ for all metric trees. We give two independent proofs of this result, first of which depends on the fact that given a metric tree M and a subset E with its diameter $2r$, for all $\epsilon > 0$ there exists $m \in M$ such that $E \subset B(m; r + \epsilon)$. The second proof depends on the calculation of underlying geometric constant, Lifschitz characteristic of a metric tree. It is well known that the classes of condensing or contractive operators defined relative to distinct measures of noncompactness are not equal in general, however in case of metric trees, we show that a map T between two metric trees is k-set contractive if and only if it is k-ball contractive for $k \geq 0$.

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1   Introduction

The study of metric trees (T-theory or $\mathbb{R}$-trees) began with J. Tits [13] in 1977 and since then, applications have been found for metric trees within many fields of mathematics. For an overview of geometry, topology, and group theory applications, consult Bestvina [4]. A complete discussion of these spaces and their relation to so-called $CAT(0)$ spaces we refer to [6]. Applications of metric trees in biology and medicine involve phylogenetic trees [12], and metric trees even find applications in computer science that involve string matching [3]. Given that metric trees go by three names there are a couple of equivalent definitions, and to understand the definition we choose to use we must first define the notion of a metric segment in a metric space.

Definition 1.1. Let $x, y \in X$, where (X, d) is a metric space. A *metric segment from x to y*, denoted $[x, y]$, is a subset of X such that $h_{x,y} : z \mapsto d(x, z)$ is an isometry from $[x, y]$ onto $[0, d(x, y)] \subset \mathbb{R}$ where $\mathbb{R}$ has the Euclidian metric.

Following the notation used for segments in $\mathbb{R}$, for a metric segment $[x, y] \subset X$ denote a half-open segment as $[x, y) := [x, y] \setminus \{y\}$ and denote an open segment from x to y as $(x, y) := [x, y] \setminus \{x, y\}$.

Definition 1.2. (M, d), a metric space, is a *metric tree* iff for all $x, y, z \in M$,

1. there exists an unique metric segment from x to y,

2. $[x, z] \cap [z, y] = \{z\} \Rightarrow [x, z] \cup [z, y] = [x, y]$.

Throughout this paper, (X, d) will be a metric space and (M, d) will be a metric tree. Furthermore, $d(x, y)$ will be denoted as xy as long as there is no fear of confusion, i.e.,

$$d(x, y) := xy.$$

Let $B(x; r) := \{y \in X \mid xy < r\}$ denote the open ball centered at x with radius r and $B_c(x; r) := \{y \in X \mid xy \leq r\}$ denote the closed ball centered at x with radius r.

Example 1.3 (A Nice Simple Tree). Let M be the subset of $\mathbb{R}^2$ where $M = [-1,1] \times \{0\} \bigcup \{1\} \times [-1,1]$. Define a metric $d : M \times M \to \mathbb{R}$ by:

$$d(x,y) = \begin{cases} \|x - y\| & \text{if } x, y \in [-1,1] \times \{0\}, \\ \|x - y\| & \text{if } x, y \in \{1\} \times [-1,1], \\ \|x - (1,0)\| + \|(1,0) - y\| & \text{otherwise.} \end{cases}$$

With this metric, (M, d) becomes a metric tree. While M looks like a tree from graph theory, this is not the case for all metric trees.

Example 1.4 (The Radial Metric). Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by:

$$d(x,y) = \begin{cases} \|x - y\| & \text{if } x = \lambda\, y \text{ for some } \lambda \in \mathbb{R}, \\ \|x\| + \|y\| & \text{otherwise.} \end{cases}$$

We can observe the d is in fact a metric and that $(\mathbb{R}^2, d)$ is a metric tree.

In order to investigate the radial metric, we will actually investigate a metric tree that is isometric to the radial metric. Let M be defined as:

$$M := \left\{ x \in \prod_{\alpha \in [0,2\pi)} \mathbb{R}_{\geq 0} \mid x_\alpha \neq 0 \text{ for at most one } \alpha \right\}.$$

Then, define a metric on M as:

$$\rho(x,y) = \sum_\alpha |x_\alpha - y_\alpha|.$$

The summation is defined, for $|x_\alpha - y_\alpha| \neq 0$ for at most two $\alpha \in [0, 2\pi)$. $(\mathbb{R}^2, d)$ and (M, ρ) are isometric by the isometry $f : \mathbb{R}^2 \to M$ with

$$f(r\, e^{i\beta})_\alpha = \begin{cases} r & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{0}$ denote the element of M that is zero in every coordinate. Observe that $f^{-1}(B(\mathbf{0}; r)_\rho) = B((0,0); r)_{Euclid}$. For $x \in M$ with $x_\beta \neq 0$, (here $f(x_\beta\, e^{i\beta}) = x$),

$$B(x; r)_\rho = f(\{t\, e^{i\beta} \mid t \in (x_\beta - r, x_\beta + r)\}) \cup B(\mathbf{0}; r - x_\beta)_\rho$$

Suppose $x^n \to x$ in (M, ρ) with $x_\beta \neq 0$. Then, there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow x^n \in B(x; x_\beta)_\rho$. Hence $n \geq N \Rightarrow x_\alpha^n = 0$ for $\alpha \neq \beta$ and $x_\beta^n \in (0, 2x_\beta)$, since $B(x; x_\beta)_\rho = f(\{t\, e^{i\beta} \mid t \in (0, 2x_\beta)\})$. In terms of the radial metric, this means that the tail end of $\{x^n\}$ is a convergent sequence in $[0, 2x_\beta] \subset \mathbb{R}$.

1.1   Properties of Metric Segments

This subsection covers some basic properties of metric segments in metric spaces drawn from Blumenthal [5]. Let $(X,\, d)$ be a metric space and for $a, b \in X$ let $S_{a,b}$ denote a metric segment from a to b (since metric segments are not necessarily unique).

Definition 1.5. Let $x, y, z \in X$. y is *between x and z*, denoted xyz, iff $xz = xy + yz$.

Remark 1.6. Given metric segment $S_{x,z} \subset X$, observe that xyz for all $y \in S_{x,z}$.

Lemma 1.7 (Blumenthal, [5])**.** *Let $a, b, c, d \in X$. If abc and acd, then abd and bcd.*

We will often refer to above property of metric segments, which is about the transitivity of betweenness of points on metric segments.

Lemma 1.8 (Blumenthal, [5])**.** *Let $a, b, p \in X$ and suppose $S_{a,p}$ and $S_{p,b}$ exist in X. $S_{a,p} \cup S_{p,b} = S_{a,b}$ iff apb.*

Theorem 1.9 (Blumenthal, [5])**.** *Let X be a complete metric space, F a closed subset of X, $a, b \in F$, and $F \subset \{x \in X \mid axb\}$. Then F is a metric segment from a to b iff F has exactly one midpoint of x, y for all $x, y \in F$.*

Theorem 1.10 (Blumenthal, [5])**.** *Let $a, b \in X$ and let $F \subset X$ be an injective arc from a to b. F is a metric segment iff for all $x, y \in F$ either axy or yxb.*

1.2   Metric Segments in Metric Trees

Within this section $(M,\, d)$ will be a metric tree.

Lemma 1.11.

1. *For $x, y \in M$, $[x, y] = \{z \in M \mid xzy\}$.*

2. *For $x, y, z \in M$, $z \in [x, y]$ only if $[x, z] \subset [x, y]$.*

The proof is classical and follows directly from the results in [5].

Theorem 1.12. *If $x, y, z \in M$, then there exists $w \in M$ such that $[x, z] \cap [y, z] = [w, z]$ and $[x, y] \cap [w, z] = \{w\}$.*

Proof. Define $f : [x, z] \cap [y, z] \to \mathbb{R}$ by $v \mapsto vz$. Since $[x, z] \cap [y, z]$ is compact, there exists $w \in [x, z] \cap [y, z]$ such that $wz \geq vz$ for all $v \in [x, z] \cap [y, z]$.

By Lemma 1.11, $[w, z] \subset [x, z] \cap [y, z]$. Conversely let $a \in [x, z] \cap [y, z]$, then we know that either xaw or waz by Theorem 1.10. If xaw, then by transitivity and xwz we know that awz. Hence $aw + wz = az \leq wz$ using the maximality of wz, hence $aw = 0$ and $a = w$. Hence $a = w$ or waz, and in either case $a \in [w, z]$ by Lemma 1.11. Therefore $[x, z] \cap [y, z] = [w, z]$.

Suppose $[x, w] \cap [w, y] \neq \{w\}$. Let $w' \in [x, w] \cap [w, y] \subset [x, z] \cap [y, z]$ with $w' \neq w$. Hence $xw'w$ and xwz, and therefore by transitivity $xw'z$. By $xw'z$, the triangle inequality, $xw'w$, and xwz,

$$xz = xw' + w'z \leq xw' + w'w + wz = xw + wz = xz.$$

Therefore, $w'z = w'w + wz > wz$, which contradicts the maximality of wz.

Hence $[x, w] \cap [w, y] = \{w\}$, therefore $[x, w] \cup [w, y] = [x, y]$ and in particular $w \in [x, y]$. Let $a \in [x, y] \cap [w, z]$ and $a \neq w$. Since $a, w \in [x, y]$ by Theorem 1.10, we know without loss of generality that xaw. By transitivity, xaw and xwz implies awz. This means that $az = aw + wz > wz$, which contradicts the maximality of wz. Therefore, $[x, y] \cap [w, z] = \{w\}$. $\square$

Theorem 1.13. *Let $a, m, x, y \in M$. Suppose m is the midpoint of $[x, y]$ and $x, y \in B(a; r)$. Then $B_c(m; \frac{xy}{2}) \subset B(a; r)$.*

Proof. Let $z \in B_c(m; \frac{xy}{2})$, so $zm \leq \frac{xy}{2}$, we want to show that $za < r$.

By Theorem 1.12, exists $w \in [x, y]$ such that $[a, x] \cap [a, y] = [a, w]$. Since $w \in [x, y]$, by Theorem 1.10 we can assume without loss of generality that xmw. By transitivity, xwa and xmw imply xma. So $\frac{xy}{2} + ma = xm + ma = xa < r$ and hence $ma < r - \frac{xy}{2}$. By the above, $za \leq zm + ma < \frac{xy}{2} + (r - \frac{xy}{2}) = r$.

Hence $za < r$ and therefore $B_c(m; \frac{xy}{2}) \subset B(a; r)$. $\square$

Corollary 1.14. *If $x, y, a \in M$ with $x, y \in B(a; r)$, then $[x, y] \subset B(a; r)$.*

Proof. Let m be the midpoint of $[x, y]$. Then $[x, y] \subset B_c(m; \frac{xy}{2})$ and therefore by Theorem 1.13, $[x, y] \subset B(a; r)$. $\qquad\square$

2   Compactness in Metric Trees

Since metric trees are very similar to $\mathbb{R}$, one might hope that the Heine-Borel theorem is true in metric trees. Unfortunately this is not the case as the following example demonstrates.

Example 2.1. Returning to Example 1.4 of the radial metric on $\mathbb{R}^2$, consider the closed unit ball at the origin $B_c((0,0); 1)$. For $\alpha, \beta \in [0, 2\pi)$, we know that $e^{i\alpha} \in B_c((0,0); 1)$ and $d(e^{i\alpha}, e^{i\beta}) = 2$ for $\alpha \neq \beta$, which implies that $B_c((0,0); 1)$ is not totally bounded. Therefore, $B_c((0,0); 1)$ is not compact.

Example 2.2. In the following we consider the river metric and define a compact and noncompact tree with respect to this metric. These simple examples motivate the characterization of compactness in metric trees. Let $M = [0, 1]^2$ and define a river metric $\rho : M \times M \to \mathbb{R}_{\geq 0}$ by:

$$\rho((x_1, y_1), (x_2, y_2)) = \left\{ \begin{array}{ll} |y_1 - y_2| & x_1 = x_2 \\ y_1 + |x_1 - x_2| + y_2 & \text{otherwise} \end{array} \right.$$

Then *a compact tree C* looks like: $C = [0, 1] \times \{0\} \cup \bigcup\limits_{n=1}^{\infty} \{\frac{1}{n}\} \times [0, \frac{1}{n}]$
and *a noncompact tree A* is: $A = [0, 1] \times \{0\} \cup \bigcup\limits_{n=1}^{\infty} \{\frac{1}{n}\} \times [0, 1]$.

Definition 2.3. Let M be a metric tree. Define F, the set of *final points of M (or leaves of M)* as

$$F := \{f \in M \mid f \notin (x, y) \text{ for all } x, y \in M\}.$$

Theorem 2.4. *If M be a compact metric tree and $a \in M$, then*

$$M = \bigcup_{f \in F} [a, f]$$

Proof. It suffices to show that $M \subset \bigcup_{f \in F}[a, f]$. Let $m \in M$, define

$$R_m = \{z \in M \mid m \in [a, z]\}$$

and observe that $R_m \neq \emptyset$ since $m \in R_m$.

We will show that R_m is closed in M. Let $y \notin R_m$. Hence there exists $u \in [a, m)$ such that $[a, y] \cap [m, y] = [u, y]$. Let $\epsilon < um$ and let $b \in B(y; \epsilon)$. Hence,

$$
\begin{aligned}
ab \quad &\leq \quad ay + yb < ay + um && \text{since } yb < \epsilon < um \\
&= \quad au + uy + um = am + uy && \text{since } auy \text{ and } aum \\
&= \quad am + (my - mu) && \text{since } muy \\
&< \quad am + my - yb && \text{since } yb < \epsilon < um \\
&\leq \quad am + mb && \text{since } my \leq mb + by,
\end{aligned}
$$

and therefore $ab < am + mb$, so $b \notin R_m$. Hence $B(y; \epsilon) \subset M \setminus R_m$, and therefore R_m is closed in M. Since M is compact it follows that R_m is compact.

Define $h : R_m \to \mathbb{R}$ by $z \mapsto az$. Since R_m is compact, there exists $f \in R_m$ such that $af \geq az$ for all $z \in R_m$.

We will show that $f \in F$. Suppose that $f \notin F$, so there exists $x, y \in M$ such that $f \in (x, y)$. Since metric segments are closed under intersections, $f \in (x, y)$, and f is an end point of $[a, f]$, we have that $[a, f] \cap [x, y] = [f, v]$ where $v \in [x, y]$. Hence by switching x and y we can claim without loss of generality that $v \in [a, f]$. Therefore, $f \in [a, y]$ and since $m \in [a, f]$, by transitivity we have that $m \in [a, y]$. Hence $y \in R_m$ and $ay = af + fy > af$, since $f \neq y$. We have a contradiction with the maximally of af in R_m, therefore $f \in F$.

Hence for all $m \in M$ there exists $f \in F$ such that $m \in [a, f]$ and therefore we have proven that $M \subset \bigcup_{f \in F}[a, f]$. $\square$

Theorem 2.5. *A metric tree M is compact if and only if*

1. $M = \bigcup_{f \in F}[a, f]$ *for all $a \in M$, and*

2. $\overline{F}$ *is compact.*

Proof. That M is compact implies (1) and (2) follows from Theorem 2.4, so we just need to prove that (1) and (2) imply that M is compact.

Let $\mathfrak{U} = \{B(x_\alpha; r_\alpha)\}_\alpha$ be an open-ball cover of M. Since $\overline{F}$ is compact, we know that there exists a finite subcover of $\overline{F}$ in $\mathfrak{U}$. Let $\bigcup_{j=1}^n B(x_j, r_j) \supset \overline{F}$ be such a subcover. If $M \subset \bigcup_{j=1}^n B(x_j, r_j)$, then we are done, so suppose there exists $a \in M$ such that $a \notin \bigcup_{j=1}^n B(x_j, r_j)$.

Let $f, f' \in F \cap B(x_j, r_j)$. Recall that there exists $c \in [f, f']$ such that $[a, f] \cap [a, f'] = [a, c]$ and that $[f, f'] \subset B(x_j, r_j)$. It follows then that $[a, f] \setminus B(x_j, r_j) = [a, f'] \setminus B(x_j, r_j)$.

For each j, let $f_j \in F \cap B(x_j, r_j)$. It follows from the above that

$$\bigcup_{f \in F \cap B(x_j, r_j)} [a, f] = [a, f_j] \cup B(x_j, r_j).$$

Then by (1) and the above claim we have that

$$M = \bigcup_{f \in F} [a, f] = \bigcup_{j=1}^n [a, f_j] \cup B(x_j, r_j) = \left(\bigcup_{j=1}^n [a, f_j] \right) \cup \left(\bigcup_{j=1}^n B(x_j, r_j) \right).$$

Since each $[a, f_j]$ is compact, we can find a finite subcover for it in $\mathfrak{U}$, call it $\mathfrak{U}_j$. Then we have that $\{B(x_j, r_j) \mid 1 \leq j \leq n\} \cup \bigcup_{j=1}^n \mathfrak{U}_j$ is a finite subcover of M in $\mathfrak{U}$. Therefore, M is compact and the theorem is proven. □

3 Measures of Noncompactness in Metric Trees

The notion of the measure of noncompactness of a subset of a metric space was introduced by Kuratowski [11] as a way to generalize Cantor's intersection theorem. In 1955, Darbo [7] applied measures of noncompactness to prove a powerful fixed point theorem and since then measures of noncompactness have been a standard notion in fixed point theory. A *set-contraction* is a mapping under which the image of any set is, in some definite sense, more compact than the set itself. The class of set (or ball)-contractions (or condensing) mappings has many applications in the study of nonlinear operators. This class is defined via the notion of measure of noncompactness of a set (mnc for brevity). A continuous map T is said to be *k-set contraction* if there is a number $k \geq 0$ such that

$$\alpha(T(D)) \leq k\alpha(D)$$

for all bounded sets D. Every compact T is a k-set contraction with $k = 0$. Compactness plays an essential role in the proof of Schauders fixed point theorem, however there are some situations where the operators are not compact. Darbo [7] utilized the concept of set-contractions by proving a generalized version of the Schauders fixed point theorem. Besides its use in fixed point theory, mncs are also a useful tool in studying theory of functional equations, including partial differential and integral equations, optimal control theory, etc. Although all measures of noncompactness are equivalent in a topological sense, the contractive constants of the mappings are not preserved when different measures are considered. Moreover, these measures are closely related to the geometrical properties of the underlying space. Therefore it is important to know the relationships between different mncs. In the following we introduce several mncs in the context of metric trees and study the relationship between them. For fixed point theorems for metric trees we refer to [1], [8] and [9].

Definition 3.1. Given metric space X and bounded subspace A, define the *set (Kuratowski) measure of noncompactness* as

$$\alpha(A) := \inf \left\{ b > 0 \mid A \subset \bigcup_{j=1}^{n} E_j \text{ for some } E_j \subset A,\, diam(E_j) \leq b \right\}.$$

Definition 3.2. Given metric space X and bounded subset A, define the *ball (Hausdorff) measure of noncompactness* as

$$\beta(A) := \inf \left\{ b > 0 \mid A \subset \bigcup_{j=1}^{n} B(x_j; b) \text{ for some } x_j \in X \right\}.$$

Lemma 3.3 (Banas and Goebel, [2])**.** *For a bounded set A of a metric space X,*
$$\beta(A) \leq \alpha(A) \leq 2\beta(A).$$

Before proceeding to prove statements about measures of noncompactness we will point out the following behavior of α and β under isometries. We omit the straightforward proofs.

Theorem 3.4. *If (X, d) and (Y, ξ) are metric spaces, $f : X \to Y$ an isometric embedding, and $A \subset X$ bounded, then $\alpha(f(A)) = \alpha(A)$.*

Lemma 3.5. *If (X, d) and (Y, ξ) are metric spaces, $f : X \to Y$ is an isometric embedding, and $A \subset X$ is bounded, then it is not necessarily the case that $\beta(f(A)) = \beta(A)$.*

Proof. Let (X, d) be a discrete metric space with an infinite number of points. Since $X = B(x; 1 + \epsilon)$ for all $\epsilon > 0$ and $\{x\} = B(x; 1)$, we know $\beta(X) = 1$.

$Y = X \cup \{x^\star\}$ where $x^\star \notin X$. Make (Y, ξ) a metric space by defining ξ by

$$\xi(a, b) = \begin{cases} 1 & \text{if } a, b \in X \\ \frac{1}{2} & \text{if exactly one of } a, b \text{ is } x^\star \end{cases}$$

Consider the isometric embedding $f : X \to Y$ by $f(x) = x$. Since $f(X) \subset B(x^\star; \frac{3}{4})$, we know that $\beta(f(X)) < 1 = \beta(X)$. Therefore, $\beta(f(X)) \neq \beta(X)$. $\qquad\square$

By the above Lemma we see that β is not invariant under isometric embeddings, however, it is not difficult to prove β is invariant under bijective isometries.

We will now turn our attention back to measures of noncompactness in metric trees.

Lemma 3.6. *Let E be a subset of metric tree M with $diam(E) = 2r$. Then for all $\epsilon > 0$ there exists $m \in M$ such that $E \subset B(m; r + \epsilon)$.*

Proof. For all $\epsilon > 0$, there exists $x, y \in E$ such that $xy > 2r - 2\epsilon$ and let m be the midpoint of $[x, y]$.

Let $z \in E$, then by Theorem 1.12, there exists $w \in [x, y]$ such that $[z, x] \cap [z, y] = [z, w]$. Without loss of generality we can assume that $m \in [w, x]$ and hence $w \in [z, x]$ by transitivity. Next, $diam(E) = 2r$ and $w \in [z, x]$ imply that

$$2r \geq zx = zm + mx = zm + \frac{xy}{2} > zm + (r - \epsilon),$$

which implies $r + \epsilon > zm$. Therefore $E \subset B(m; r + \epsilon)$. $\qquad\square$

Theorem 3.7. *Let A be a bounded subset of metric tree M. Then $\alpha(A) = 2\beta(A)$.*

Proof. By Lemma 3.3 it suffices to prove that $\alpha(A) \geq 2\beta(A)$.

Let $2r > \alpha(A)$, then there exists $E_j \subset M$ such that $diam(E_j) \leq 2r$ and $A \subset \bigcup_{j=1}^n E_j$. By Lemma 3.6, for all $\epsilon > 0$ there exists $m_j \in E_j$ such that $E_j \subset B(m_j; r + \epsilon)$. Hence $A \subset \bigcup_{j=1}^n B(m_j; r + \epsilon)$, so $\beta(A) \leq r + \epsilon$ for all $\epsilon > 0$. Therefore, $\beta(A) \leq r$.

So $\beta(A) \leq r$ for all r such that $2r > \alpha(A)$. Therefore $\alpha(A) \geq 2\beta(A)$. $\square$

Remark 3.8. Some authors define the *ball measure of noncompactness* as:

$$\beta^*(A) := \inf \left\{ b > 0 \mid A \subset \bigcup_{j=1}^n B(x_j; r_j) \text{ with } diam(B(x_j, r_j)) \leq b \right\}.$$

Theorem 3.9. *If A be a bounded subset of a metric tree M, then $\beta^*(A) = 2\beta(A)$.*

Proof. Every ϵ-diameter ball cover is a ϵ-diameter set cover, so therefore $\alpha(A) \leq \beta^*(A)$. Hence by Theorem 3.7, $2\beta(A) \leq \beta^*(A)$.

Conversely since every ϵ-radius ball cover is a 2ϵ-diameter ball cover, $\beta^*(A) \leq 2\beta(A)$. Therefore $\beta^*(A) = 2\beta(A)$. $\square$

Definition 3.10. Let $T : X \to Y$ be a continuous map between metric spaces.

For $k \geq 0$, T is *k-set-contractive* iff for every bounded subset $A \subset X$, that $\alpha(T(A)) \leq k \, \alpha(A)$. Analogously, T is *k-ball-contractive* iff for all bounded subsets $A \subset X$, $\beta(T(A)) \leq k \, \beta(A)$.

T is *set-condensing (ball-condensing)* iff T is k-set-contractive (k-ball contractive) with $k < 1$.

Lemma 3.11. *Let $T : X \to Y$ be a continuous map between metric spaces. If T is k-set-contractive, then T is 2k-ball-contractive. If T is k-ball-contractive, then T is 2k-set-contractive.*

Proof. Suppose T is k-set-contractive and let $A \subset X$ be bounded. Then by Lemma 3.3, we know the following:

$$\beta(T(A)) \leq \alpha(T(A)) \leq k\,\alpha(A) \leq 2k\,\beta(A).$$

Therefore, T is $2k$-ball-contractive.

Analogously, suppose T is k-ball-contractive and let $A \subset X$ be bounded. Then by Lemma 3.3, we know the following:

$$\alpha(T(A)) \leq 2\,\beta(T(A)) \leq 2k\,\beta(A) \leq 2k\,\alpha(A).$$

Therefore T is $2k$-set-contractive. $\qquad\square$

Theorem 3.12. *Let M, N be metric trees and $D \subset M$. Then T is k-set-contractive iff T is k-ball-contractive.*

Proof. Let $A \subset D$ be bounded.

Suppose that T is k-set-contractive, then by Theorem 3.7,

$$\beta(T(A)) = \frac{1}{2}\,\alpha(T(A)) \leq \frac{k}{2}\,\alpha(A) = k\,\beta(A).$$

Therefore T is k-ball-contractive.

Suppose that T is k-ball-contractive, then by Theorem 3.7,

$$\alpha(T(A)) = 2\,\beta(T(A)) \leq 2k\,\beta(A) = k\,\alpha(A).$$

Therefore T is k-set-contractive. $\qquad\square$

We will now examine the Lifschitz characteristic of a metric space and its connection to measures of noncompactness, (see [10]).

Definition 3.13. For a metric space X, $b \in \mathbb{R}_{>0}$ is *Lifschitz for X* iff there exists $a > 1$ such that for all $x, y \in X, r > 0$, if $xy > r$, then there exists $z \in X$ such that $B_c(x; ar) \cap B_c(y; br) \subset B_c(z; r)$.

Definition 3.14. For a metric space X, define the *Lifschitz characteristic of X* as

$$\kappa(X) := \sup\{b > 0 \mid b \text{ is Lifschitz for } X\}.$$

Remark 3.15. Observe that for any metric space X, that $\kappa(X) \geq 1$. For if $b \leq 1$, then by choosing $z = y$ we can see for any $a > 1$ that, $B_c(x; ar) \cap B_c(y; br) \subset B_c(z; r)$.

Theorem 3.16. *Let M be a metric tree, then $\kappa(M) = 2$.*

Proof of $\kappa \geq 2$. We will show that every $0 < b < 2$ is Lifschitz for M. Let $b = 2 - 2\epsilon$, where $0 < \epsilon < 1$ and select $a > 1$ to be $a = 1 + \epsilon$. For every $r > 0$ and $x, y \in M$ with $xy > r$, let $z \in [x, y]$ such that $xz = \epsilon r$.

Suppose that $w \in B_c(x; ar) \cap B_c(y; br)$. By Theorem 1.12, there exists $u \in [x, y]$ such that $[w, x] \cap [w, y] = [w, u]$. Since $z \in [x, y]$ too, we know that $z \in [x, u]$ or $z \in [u, y]$.

If $z \in [x, u]$, then by $u \in [x, w]$ and transitivity, we know $z \in [x, w]$. Hence,

$$
\begin{aligned}
zw &= xw - xz \leq ar - xz && \text{by } z \in [x, w] \text{ and } w \in B_c(x; ar), \\
&= (r + \epsilon r) - \epsilon r && \text{by } a = 1 + \epsilon \text{ and } xz = \epsilon r, \\
&= r.
\end{aligned}
$$

If $z \in [u, y]$, then by $u \in [w, y]$ and transitivity, we know $z \in [w, y]$. Hence,

$$
\begin{aligned}
zw &= wy - zy && \text{by } z \in [w, y] \\
&= wy - (xy - xz) && \text{by } z \in [x, y], \\
&< br - r + \epsilon r && \text{by } w \in B_c(y; br),\ r < xy,\ \text{and } xz = \epsilon r, \\
&= (2r - 2\epsilon r) - r + \epsilon r && \text{by } b = 2 - 2\epsilon, \\
&= r - \epsilon r < r.
\end{aligned}
$$

We know that $z \in [x, u]$ or $z \in [u, y]$, and in either case $zw \leq r$, so $w \in B_c(z; r)$. Therefore $B_c(x; ar) \cap B_c(y; br) \subset B_c(z; r)$. Hence b is Lifschitz for M for all $b < 2$ and therefore $\kappa(M) \geq 2$. $\qquad\square$

Proof of $\kappa(M) \leq 2$. Suppose for some metric tree M, that b is Lifschitz for M. Hence there exists $a > 1$ such that for all $r > 0$ and $x, y \in M$ with $xy > r$, there exists $z \in M$ such that $B_c(x; ar) \cap B_c(y; 2r) \subset B_c(z; r)$.

Let $w, v \in M$ with $wv = 4r$ for some $r > 0$ and let $a > 1$. Let y be the midpoint of $[w, v]$ and let $x \in [y, v]$ with $r < yx < ar$. Hence $[w, v] \subset B_c(y; 2r)$ and $[u, v] \subset B_c(x; ar)$ where $u \in [w, v]$ with $ux = ar$. Therefore, $[u, v] \subset B_c(x; ar) \cap B_c(y; 2r)$.

However, $diam([u,v]) = uy + yv > yv = 2r$. So for all $z \in M$, $[u,v] \nsubseteq B_c(z;r)$ since $diam(B_c(z;r)) \leq 2r$. Therefore for all $z \in M$, $B_c(x;ar) \cap B_c(y;2r) \nsubseteq B_c(z;r)$, which contradicts that $b = 2$ is Lifschitz for M. Therefore, for all metric trees M, $\kappa(M) \leq 2$. $\qquad\square$

Lemma 3.17 (Webb and Zhao, [14])**.** *Let M be a metric space and $A \subset M$, with $diam(A) = d$. For all $0 < b < \kappa(M)$, there exists $z \in M$ such that $A \subset B_c(z, \frac{d}{b})$.*

Theorem 3.18 (Webb and Zhao, [14])**.** *Let A be a bounded subset of metric space M, then $\kappa(M)\beta(A) \leq \alpha(A)$.*

Proof. By the definition of $\alpha(A)$, for all δ with $0 < \delta < \kappa(M)$, there exists $E_j \subset M$ such that $diam(E_j) \leq \alpha(A) + \delta$ and $A \subset \bigcup_{j=1}^n E_j$.

By Lemma 3.17, for $b = \kappa(M) - \delta$ and each E_j, there exists $z_j \in M$ such that $E_j \subset B_c(z_j, \frac{\alpha(A)+\delta}{\kappa(M)-\delta})$. Hence,

$$A \subset \bigcup_{j=1}^n E_j \subset \bigcup_{j=1}^n B_c\left(z_j, \frac{\alpha(A)+\delta}{\kappa(M)-\delta}\right).$$

So for all δ such that $0 < \delta < \kappa(M)$, we have that $\beta(A) \leq \frac{\alpha(A)+\delta}{\kappa(M)-\delta}$. Therefore, $\kappa(M)\beta(A) \leq \alpha(A)$. $\qquad\square$

Remark 3.19. Theorems 3.16 and 3.18, show that $2\beta(A) \leq \alpha(A)$ for any bounded subset A in a metric tree M. Therefore we have an alternate proof that for any bounded subset A in a metric tree M, $\alpha(A) = 2\beta(A)$. Moreover, by Theorems 3.4 and 3.5 we know that α is invariant under isometric embedding $f : X \to Y$ and β is not. However if X and Y are metric trees, β will be invariant under f as well, since $\alpha(A) = 2\beta(A)$.

References

[1] A.G. Aksoy and M. A. Khamsi, *A Selection Theorem in Metric Trees*, Proc. Amer. Math. Soc. **134** (2006), 2957–2966.

[2] J. Banas and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Appl. Math, vol. 60, Dekker, New York, 1980.

[3] I. Bartolini, P. Ciaccia, and M. Patella, *String Matching with Metric Trees Using Approximate Distance*, SPIR, Lecture Notes in Computer Science, Springer-Verlag, vol. 2476, 2002, 271–283.

[4] M. Bestvina, $\mathbb{R}$-*trees in Topology, Geometry, and Group Theory*, Handbook of geometric topology, North-Holland, Amsterdam, 2002, 55–91.

[5] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Oxford University Press, London, 1953.

[6] M. Bridson and A. Haefliger, *Metric Spaces of Nonpositive Curvature*, Grundlehren der Mathematischen Wissenschaften, vol. 319, Springer-Verlag, Berlin, 1999.

[7] G. Darbo, *Punti Uniti in Trasformazioni a Codominio Non Compatto*, Rend. Sem. Mat. Univ. Padova **24** (1955), 84–92.

[8] R. Espinola and W. A. Kirk, *Fixed Point Theorems in $\mathbb{R}$-trees with Applications to Graph Theory*, Topology Appl. **153** (2006), 1046–1055.

[9] W. A. Kirk, *Fixed Point Theorems in* CAT(0) *Spaces and $\mathbb{R}$-Trees*, Fixed Point Theory Appl. (2004), 309–316.

[10] M. A. Krasnosel'skii and P. P. Zabreiko, *Geometric Methods of Nonlinear Analysis*, Springer, Berlin, 1984.

[11] K. Kuratowski, *Sur les Espaces Complets*, Fund. Math. **15** (1930), 301–309.

[12] C. Semple and M. Steel, *Phylogenetics*, Oxford Lecture Series in Mathematics and its Applications, vol. 24, 2003.

[13] J. Tits, *A Theorem of Lie-Kolchin for Trees*, Contributions to Algebra: A Collection of Papers Dedicated to Ellis Kolchin, Academic Press, New York, 1977.

[14] J.R.L. Webb and W. Zhao, *On Connections Between Set and Ball Measures of Noncompactness*, Bull. London Math. Soc. **22** (1990), 471–477.

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